

Wigner Random Matrices with Non-Symmetrically Distributed Entries

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Received February 7, 2007; accepted April 30, 2007

Published Online: June 1, 2007

We show that the spectral radius of an $N \times N$ random symmetric matrix with i.i.d. bounded centered but non-symmetrically distributed entries is bounded from above by $2\sigma + o(N^{-6/11+\varepsilon})$, where σ^2 is the variance of the matrix entries and ε is an arbitrary small positive number. Our bound improves the earlier results by Z. Füredi and J. Komlós (1981), and Van Vu (2005).

KEY WORDS: Wigner random matrices, largest eigenvalue

1. MODEL

We consider random symmetric matrices with i.i.d. centered but non-symmetrically distributed entries above the diagonal. To be more precise, let μ be a probability distribution with compact support K such that

$$\int_{\mathbb{R}} x d\mu = 0, \quad \int_{\mathbb{R}} x^2 d\mu = \sigma^2, \quad \int_{\mathbb{R}} x^3 d\mu = \mu_3 \neq 0 \quad \text{and} \\ \int_{\mathbb{R}} |x|^k d\mu \leq K^k, \quad \forall k \geq 4. \quad (1)$$

Consider a sequence of random symmetric matrices

$$A_N = \frac{1}{\sqrt{N}}(a_{ij})_{i,j=1}^N,$$

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where the $a_{ij}, i \leq j$ are i.i.d. random variables with distribution μ . The scope of this paper is to investigate the limiting spectral radius of the random matrix A_N as N goes to infinity.

To obtain an upper bound on the spectral radius of A_N , we compute the asymptotics of expectation of traces of high powers of A_N :

$$\mathbb{E}[\text{Tr} A_N^{2s_N}], \quad \text{where } s_N \rightarrow \infty \text{ as } N \rightarrow \infty. \tag{2}$$

1.1. Results

The main result of the paper is the following

Theorem 1.1. *Let λ_{\max} be the largest eigenvalue of the matrix A_N and $\varepsilon > 0$. Then*

$$\lambda_{\max} \leq 2\sigma + o(N^{-6/11+\varepsilon}) \tag{3}$$

with probability going to 1 as $N \rightarrow \infty$.

Remark 1.1. A similar result holds in the Hermitian case. Since the proof is essentially the same, we will discuss only the real symmetric case in this paper. Our result also holds true if one replaces the largest eigenvalue of A_N by its spectral norm $\|A_N\| = \max_i |\lambda_i|$.

Theorem 1.1 is a simple corollary of the following technical result. Let us denote by M_N the matrix $(a_{ij})_{i,j=1}^N$.

Proposition 1.1. *Assume that $s_N = O(N^{1/2+\eta})$ where $\eta < 1/22$. Then*

$$\mathbb{E}[\text{Tr} M_N^{2s_N}] = \mathbb{E}[\text{Tr} W_N^{2s_N}](1 + o(1)),$$

where W_N is a standard Wigner matrix with symmetrically distributed sub-Gaussian entries of variance σ^2 .

The asymptotics of $\mathbb{E}[\text{Tr} W_N^{2s_N}]$ was calculated in Refs. 10–12. In particular,

$$\mathbb{E}[\text{Tr} W_N^{2s_N}] = N^{s_N+1} T_{0,2s_N} \sigma^{2s_N} (1 + o(1)) = \frac{N^{s_N+1}}{\pi^{1/2} s_N^{3/2}} (2\sigma)^{2s_N} (1 + o(1)). \tag{4}$$

as long as $s_N = o(N^{2/3})$. In (4), $T_{0,2s}$ is the famous Catalan number, counting the number of possible trajectories of a simple random walk of length $2s$ in the positive quadrant that return to the origin. Such trajectories are also known as Dyck paths.

A standard application of the Markov inequality then derives the upper bound (3) from Proposition 1.1. since

$$\mathbb{E}(\lambda_{\max})^{2s_N} \leq \mathbb{E}\|A_N\|^{2s_N} \leq \mathbb{E}[\text{Tr } A_N^{2s_N}].$$

We note that the leading term 2σ in 3 is the right edge of the Wigner semicircle law.^{2,16,17!}

Theorem 1.1 strengthens upper bounds on the largest eigenvalue of Wigner random matrices with non-symmetrically distributed entries obtained earlier by Füredi and Komlós⁽⁴⁾ and Vu.⁽¹⁵⁾ We recall that in Ref. 4 the authors established that $\lambda_{\max} \leq 2\sigma + O(N^{-1/6} \ln N)$, and recently Vu⁽¹⁵⁾ improved the upper bound to $\lambda_{\max} \leq 2\sigma + O(N^{-1/4} \ln N)$. It was shown by Guionnet and Zeitouni,⁽⁵⁾ and Alon, Krivelevich, and Vu⁽¹⁾ by applying the concentration of measure technique that the largest eigenvalue is strongly concentrated around its mean. Namely (see Ref. 6)

$$\mathbb{P}(|\lambda_{\max} - \mathbb{E}(\lambda_{\max})| \geq KtN^{-1/2}) \leq 4e^{-t^2/32}, \tag{5}$$

where K is the uniform upper bound of the matrix entries $\{a_{ij}\}$ from (1). Using the technique presented in this paper, one can also obtain a lower bound on the spectral norm of A_N . Namely, we show in Ref. 9 that for any positive $\varepsilon > 0$ one has the lower bound $\|A_N\| \geq 2\sigma - N^{-6/11+\varepsilon}$, with probability going to 1 as $N \rightarrow \infty$.

More is known if the matrix entries of a Wigner matrix are sub-Gaussian and have symmetric distribution. Then the largest eigenvalue deviates from the soft edge 2σ on the order $O(N^{-2/3})$ and the limiting distribution of the rescaled largest eigenvalue can be shown⁽¹²⁾ to obey Tracy-Widom law⁽¹⁴⁾:

$$\lim_{N \rightarrow \infty} \mathbb{P}(\lambda_{\max} \leq 2\sigma + \sigma x N^{-2/3}) = \exp\left(-1/2 \int_x^\infty q(t) + (t-x)q^2(t) dt\right),$$

where $q(x)$ is the solution of the Painlevé II differential equation $q''(x) = xq(x) + 2q^3(x)$ with the asymptotics at infinity $q(x) \sim Ai(x)$ as $x \rightarrow +\infty$. It is reasonable to expect that in the non-symmetric case, the largest eigenvalue will have the Tracy-Widom distribution in the limit as well. However, at this moment this question is beyond the reach of our technique.

1.2. Sketch of the Proof

To investigate the leading term in the asymptotic expansion of (2), we use the combinatorial machinery developed for the standard Wigner random matrices with symmetrically distributed entries. Writing down the trace of $A_N^{2s_N}$ in terms of

the matrix entries of A_N , one obtains that

$$\mathbb{E}[Tr A_N^{2s_N}] = \sum_{i_0, i_1, \dots, i_{2s_N-1}} \mathbb{E} \left[\prod_{j=0}^{2s_N-1} \frac{a_{i_j i_{j+1}}}{\sqrt{N}} \right], \tag{6}$$

where we use the convention that $i_{2s_N} = i_0$. We associate a path \mathcal{P} on the set of N vertices $\{1, 2, \dots, N\}$ to each term in the expansion of (2) as follows

$$\mathcal{P} = i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{2s_N-1} \rightarrow i_{2s_N} = i_0. \tag{7}$$

As the entries a_{ij} are centered, for a term in the above sum (6) to yield a non zero contribution, all its (non-oriented) edges must appear at least twice. Due to the fact that the entries are not symmetrically distributed, such a path can admit edges which appear an odd number of times. By the above remark, only the paths with odd edges appearing at least three times have to be taken into account. Clearly, a path of even length must have an even number of odd edges. Let us denote the number of odd edges by $2l$.

The contribution of even paths (no odd edges) is known from the results established by Ya. Sinai and one of the authors in Refs. 10, 11. The combinatorial technique presented in these papers was further extended in Refs. 7, 8, 12, 13.

Before considering the combinatorics, we start with a few preliminary definitions.

Definition 1.1. A closed path is a sequence of edges $\mathcal{P} = \{(i_0, i_1), (i_1, i_2), \dots, (i_{s_N-1}, i_{s_N})\}$ starting and ending with the same vertex (i.e. $i_{s_N} = i_0$). A path admitting at least one odd edge is called an odd path.

Definition 1.2. When a (non-oriented) edge appears in a path \mathcal{P} an odd number of times, we call its last occurrence a *non closed edge* or a *non-returned edge*.

Definition 1.3. The instant j is said to be marked for the closed path \mathcal{P} if a non-oriented (i_{j-1}, i_j) occurs in \mathcal{P} an odd number of times up to the moment j (included). The other instants are said to be unmarked.

Remark 1.2. It is possible to show that one can use the technique of Ref. 10 to obtain a polynomial upper bound on $\mathbb{E}[Tr A_N^{2s_N}]$ for $s_N \leq \text{Const}N^{1/4}$ thus recovering the upper bound

$$\lambda_{\max} \leq 2\sigma + O(N^{-1/4} \ln N) \tag{8}$$

obtained in Ref. 15. To show this, we start with the path \mathcal{P} from (7) and construct a new path $\tilde{\mathcal{P}}$ in the following way. The new path $\tilde{\mathcal{P}}$ will be a closed even path of length $2s_N + 2l$ on the set of $N + 1$ vertices $\{1, 2, \dots, N + 1\}$. We keep all edges that are not non-returned edges of \mathcal{P} exactly as they appear in \mathcal{P} . All together,

there are $2s_N - 2l$ instances of time corresponding to the edges that are not non-returned. In addition, there are $2l$ instances corresponding to non-returned edges. These $2l$ instances correspond to the last occurrences of odd edges. Suppose for example that at moment $0 < j \leq 2s_N$ an odd edge (i_j, i_{j+1}) appears for the last time. Then in the path $\tilde{\mathcal{P}}$, we replace the edge (i_j, i_{j+1}) with two edges $(i_j, N + 1)$ and $(N + 1, i_{j+1})$. We do the same thing for all $2l$ non-returned edges. It is not difficult to see that the set of (non-oriented) non-returned edges can be viewed as a union of cycles. Therefore, each vertex appears an even number of times as an end point of a non-returned edge. One can show then that the path $\tilde{\mathcal{P}}$ is an even closed path, and it has at least $2l$ self-intersections. We conclude that (6) can be bounded from above by

$$(\text{const}N)^l \sum_{i_0, i_1, \dots, i_{2s_N+2l-1}}^* \mathbb{E} \left[\prod_{j=0}^{2s_N+2l-1} \frac{a_{i_j i_{j+1}}}{\sqrt{N}} \right], \tag{9}$$

where the sum in (9) is restricted only to closed even paths with at least $2l$ self-intersections. It was shown in Ref. 10 that the sum \sum^* is bounded from above by

$$\frac{(s_N^2/N)^{2l}}{(2l)!} \mathbb{E} [Tr W_{N+1}^{2s_N+2l}].$$

The bound implies that

$$\mathbb{E}(\lambda_{\max}^{2s_N}) \leq \mathbb{E}[Tr A_N^{2s_N}] \leq \text{const} \frac{N}{s_N^{3/2}} \exp(\text{const}_N^2/n^{1/2}) \tag{10}$$

and the bound (8) follows by applying the Markov inequality.

In this paper, we mainly concentrate on the contribution of paths that admit odd edges. Note that due to Assumption (1), each path contributing to (6) admits an even number of odd edges. The idea of the proof is to notice that a path of length $2s$ with $2l > 1$ odd edges can be obtained from an even ‘‘path’’ \mathcal{P}' (which could be a single closed even path or a collection of several closed paths) of length $2s - 2l$ by inserting at some moments of time the unreturned edges (see Definition 2.2 below), chosen amongst the edges of \mathcal{P}' . The contribution of non-even paths can then be estimated from the contribution of even paths of smaller length. We then use the asymptotics established in Refs. 10, 11 to study their contribution to (2). As the reader will see, the arguments presented in this paper are somewhat simpler in the case $s_N = o(\sqrt{N})$ which is presented in Sec. 3 (the proof of the Proposition 1.1 in this regime implies the upper bound $\lambda_{\max} \leq 2\sigma + o(N^{-1/2+\epsilon})$ for any arbitrary small $\epsilon > 0$). The case of greater scales requires some additional ideas presented in Sec. 4.

2. FROM AN ODD PATH TO AN EVEN PATH

In this section, we define a procedure which, starting from a path \mathcal{P} of length $2s$ with $2l$ odd edges, associates a new ‘‘path’’ \mathcal{P}' . In general, \mathcal{P}' will not be a single path but rather a sequence of paths. Nevertheless, it will be convenient to think about \mathcal{P}' as a path. \mathcal{P}' will be of length $2s - 2l$ and will have the same edges as \mathcal{P} , except that the last occurrence of each odd edge will be removed. As a result, each edge will appear in \mathcal{P}' an even number of times.

2.1. Description of the Gluing Procedure

Consider a path \mathcal{P} of length $2s$ and with $2l$ non-returned edges. The set of the moments of the last occurrences of the odd edges is, by definition, a subset of $\{1, 2, \dots, 2s\}$, and we can view it as a union of J disjoint non-empty intervals on the integer lattice, $1 \leq J \leq 2l$. As a result, we split the set of the odd edges into $1 \leq J \leq 2s$ disjoint subsequences. We denote these subsequences by $S_i, i = 1, \dots, J$. Let also e_i (resp. f_i) be the left (resp. right) endpoint of S_i and set $f_0 = e_{J+1} = i_0$ where i_0 is the origin of the path \mathcal{P} . Finally, define $J + 1$ subpaths of \mathcal{P} as follows. Let $\mathcal{P}_i, i = 0, \dots, J$ be the subpath starting at f_i and ending at e_{i+1} . Now, we are going to show that we can reorder the \mathcal{P}_i 's in such a way that we obtain a succession of subpaths. The following result is a basic fact, which we state as a lemma.

Lemma 2.1. *For any $i = 1, \dots, J$, there exists $i' \in [1, J]$ such that $e_i = e_{i'}$ or $e_i = f_{i'}$.*

We choose the way to reorder the subpaths $\mathcal{P}_0, \dots, \mathcal{P}_J$ as follows. At this point, it is useful to associate to the set of the subpaths $\mathcal{P}_i, 0 \leq i \leq J$ a graph G on the set of vertices $\mathcal{L} = \{e_i, f_i, i = 0, \dots, J\}$. G is built as follows. We draw an edge between two vertices $v_i, v_j \in \mathcal{L}$ if there exists a subpath \mathcal{P}_k admitting v_i and v_j as the end points. Denote by $1 \leq I' \leq J$ the number of connected components of G . It is a basic fact in the Graph Theory that we could glue the subpaths \mathcal{P}_i associated to the same connected component of G without raising a pen. Yet, we do not impose such a restriction in the gluing procedure and consider all possible gluings.

Let us consider the subpaths associated to the vertices of the connected component of i_0 in the order they are read in \mathcal{P} . We first read \mathcal{P}_0 . By the definition of \mathcal{P}_0 , the right end point of \mathcal{P}_0 is e_1 . We then choose another subpath \mathcal{P}_{i_1} which also has e_1 as an end point. The existence of such a path follows from Lemma 2.1. We glue these two subpaths in the following way. We read the edges of \mathcal{P}_{i_1} in the reverse direction if e_1 is the right end point of \mathcal{P}_{i_1} or in the forward direction otherwise. Call $\mathcal{P}_0 \cup \mathcal{P}_{i_1}$ the subpath obtained. To iterate the procedure, we now

look for a path $\mathcal{P}_{i_2}, i_2 \neq 0, i_1$ one of which end points coincides with the right end point of $\mathcal{P}_o \cup \mathcal{P}_{i_1}$. We then glue \mathcal{P}_{i_2} to $\mathcal{P}_o \cup \mathcal{P}_{i_1}$ in the same way as explained above and obtain the subpath $\mathcal{P}_o \cup \mathcal{P}_{i_1} \cup \mathcal{P}_{i_2}$. We keep gluing the subpaths until we obtain the subpath $\mathcal{P}_o \cup \mathcal{P}_{i_1} \cup \dots \cup \mathcal{P}_{i_k}, 1 \leq k \leq J - 1$ which is a closed path (i.e. its terminal point coincides with the starting point i_0). At this moment, we stop the procedure and start a new gluing as follows. If i_0 occurs as an end point of some subpath \mathcal{P}_j which has not been glued yet, we read the subpath \mathcal{P}_j in such a direction that its starting point is i_0 and we start a new gluing procedure with this subpath. Otherwise, we consider the first \mathcal{P}_i which has not yet been glued. An important observation is that its left end point has necessarily occurred in $\mathcal{P}_o \cup \mathcal{P}_{i_1} \cup \dots \cup \mathcal{P}_{i_k}$, due to the fact that there exists a sequence of odd edges in \mathcal{P} leading to this vertex and starting from one of the endpoints of \mathcal{P}_o , or \mathcal{P}_{i_1}, \dots or \mathcal{P}_{i_k} . We iterate the gluing procedure starting with \mathcal{P}_i . We use the same procedure for all connected components of G . As a result of the gluing procedure described above, we end up with a sequence of $I_0 \geq I'$ paths, denoted $\tilde{W}_i, 0 \leq i \leq I_0 - 1$ with origins $v_{i_j} \in \mathcal{L}, 0 \leq j \leq I_0 - 1, v_{i_0} = i_0$.

Our next goal is to construct a “path” \mathcal{P}' by the concatenation of the paths \tilde{W}_i . Let us re-order the paths \tilde{W}_i arbitrarily (except that we start with \tilde{W}_0) in such a way that we first read all the paths with the origin i_0 . We call W_0 the path obtained by the concatenation of these paths. Then, we read all the paths with origin v_1 and concatenate them obtaining W_1 , and so on. As a result, we obtain a sequence of paths W_0, W_1, \dots, W_{I-1} . Finally, we concatenate these paths, and denote by \mathcal{P}' the “path” obtained by the concatenation of the $W_i, 0 \leq i \leq I - 1$. Note that \mathcal{P}' is not necessarily a real path in a sense of the Definition 2.1, since at the end of each W_i , in principle, one can switch to another vertex. Nevertheless, the order in which the paths W_i are constructed ensures that the origin of a path where such a switch happens is a marked vertex of \mathcal{P}' . Furthermore, the vertices of \mathcal{P}' corresponding to the instants of such switches are pairwise distinct.

Remark 2.1. Let us estimate the number of possible ways to glue the sub-paths \mathcal{P}_i associated to a given path \mathcal{P} . Call \mathcal{E}_i the class of vertices occurring $2i$ times as an endpoint of a sequence of odd edges in \mathcal{P} . Set $E_i := \#\mathcal{E}_i$. Then there are at least

$$\prod_{i=2}^J (i!)^{E_i} \text{const}, \quad \text{const} < 1, \tag{11}$$

possible gluings associated to a given path \mathcal{P} . Indeed, there are $(2A - 1)(2A - 3) \dots 3 \cdot 1$ possible ways to glue subpaths with a common vertex $v, v \in \mathcal{E}_A$ as an end point (we just partition the set of such subpaths into pairs). One can also note that \mathcal{P}_o necessarily starts the path and that each vertex being the origin of a W_i is glued one time less. The estimate (11) will be of importance in Sec. 4.1.1.

Remark 2.2. Actually, the order in which the W_i 's are read in \mathcal{P}' will be irrelevant in the following. The important fact is that the origin of each W_i , $i \geq 1$ is a marked vertex of \mathcal{P}' and that they are pairwise distinct. The gluing procedure can also be seen as associating a path W_0 starting with i_0 and a collection of unordered paths W_i , $i > 1$, all of which have a marked origin.

2.2. The Structure of \mathcal{P}'

In this subsection, we study in more detail the structure of \mathcal{P}' . Three cases can occur:

- Case A: the gluing procedure leads to one *real* closed even path \mathcal{P}' (in a sense of Definition 1.1).
- Case B: the gluing procedure leads to a “path” \mathcal{P}' which is really a sequence of $I \geq 2$ closed even paths with respective origins $\{i_0, v_i, 1 \leq i \leq I - 1\}$ and where each v_i is a marked vertex of the path \mathcal{P}' .
- Case C: the gluing procedure leads to a sequence of $I \geq 2$ paths, some with odd edges. In this case, the I paths also have respective origins $i_0, v_i, i \leq I - 1$, where each v_i is a marked vertex of the path \mathcal{P}' . Furthermore, the union of these paths has only even edges.

In all the cases, \mathcal{P}' is of length $2s - 2l$.

In Case C, where at least one path W_i has “odd” edges, we apply an additional gluing procedure, which glues some of the paths W_i together so that we will end up, as in the preceding case, with a sequence of closed even paths of total length $2s - 2l - 2q$ for some $q > 0$. The goal here is to show that the paths of Case C are negligible with respect to those of Case B or Case A. This part appeals to some results established in Refs. 10, 11. As the union of the paths W_i has only even edges, each edge which is odd in some W_i is also odd in some other path W_j . Here we use the *construction procedure* already used in Ref. 10 to glue the paths.

Let \tilde{i} denote the smallest index such that $W_{\tilde{i}}$ has an odd edge. Let then \tilde{e} (resp. $t_{\tilde{e}}$) be the first occurrence of an odd edge in $W_{\tilde{i}}$ (resp. the instant of the first occurrence) and $\tilde{j} > \tilde{i}$ be the smallest index such that $W_{\tilde{j}}$ has the edge \tilde{e} as an odd edge. Let also $t'_{\tilde{e}}$ be the instant of the first occurrence of \tilde{e} in $W_{\tilde{j}}$. Then, we are going to form $W_{\tilde{i}} \vee W_{\tilde{j}}$ as follows. Assume first that the occurrences of the edge \tilde{e} at instances $t_{\tilde{e}}$ in $W_{\tilde{i}}$ and $t'_{\tilde{e}}$ in $W_{\tilde{j}}$ have opposite directions. In this case, we read the first $t_{\tilde{e}} - 1$ edges of $W_{\tilde{i}}$, then switch to $W_{\tilde{j}}$ and read the edges of $W_{\tilde{j}}$ from the instant $t'_{\tilde{e}} + 1$ to the end of $W_{\tilde{j}}$. After that, we restart at the origin of $W_{\tilde{j}}$ and read all the edges of this path until (but not including) the selected occurrence of the edge \tilde{e} . At this point we switch back to $W_{\tilde{i}}$ and finish by reading its remaining edges. As a result, we obtain the path $W_{\tilde{i}} \vee W_{\tilde{j}}$ by erasing the edge \tilde{e} twice: once from $W_{\tilde{i}}$ and once from $W_{\tilde{j}}$.

If t_e and t'_e are in the same direction, the procedure is quite similar. The difference is that we then read the edges of $W_{\tilde{j}}$ in the reverse direction. We read the first $t'_e - 1$ edges of $W_{\tilde{j}}$ backwards and so on. We again end up with a path $W_{\tilde{i}} \vee W_{\tilde{j}}$ of length $l(W_{\tilde{i}}) + l(W_{\tilde{j}}) - 2$. As a result of this procedure, we replace two paths $W_{\tilde{i}}$ and $W_{\tilde{j}}$ with one path $W_{\tilde{i}} \vee W_{\tilde{j}}$. In the process, we erased two appearances of a non-oriented odd edge. We continue this algorithm until we end up with a sequence of $I - I_1$ closed even paths. If we repeat the described gluing procedure I_1 times, we erase in the process $2I_1$ appearances of odd edges. The total length of the union of the final Dyck paths obtained in this way is $2s - 2l - 2I_1$.

Let us denote these $I - I_1$ closed even paths by $D_j, j = 0, \dots, I - I_1 - 1$. They are of total length $2s - 2l - 2I_1$. To reconstruct the paths $W_i, 0 \leq i \leq I - 1$ from the paths $D_i, 0 \leq i \leq I - I_1 - 1$, one has to choose a) the moments where one erased the I_1 edges, one of which we denoted above by \tilde{e} , b) the lengths, and c) the origins of the I_1 paths corresponding to the instants of switch. A trivial upper bound for the number of preimages $\{W_i, i = 0, \dots, I - 1\}$ of these $I - I_1$ Dyck paths is

$$\binom{2s}{I_1} (4s)^{I_1} (2s)^{I_1}.$$

Now due to the fact that such a choice of the origins, lengths and instants of switch of the glued paths determines the odd edges glued pairwise, the weight of the $I - I_1$ Dyck paths is multiplied by a factor of order $(\text{const}/N)^{I_1}$. Therefore, the number of preimages times the multiplying factor $(\text{const}/N)^{I_1}$ is at most of order

$$\binom{2s}{I_1} \times \left(\frac{\text{const} \times s^2}{N} \right)^{I_1} \ll \binom{2s}{I_1} \quad \text{if } s \ll \sqrt{N}. \tag{12}$$

One then can use this estimate below in Sec. 3.1.2, formula (28) to show that such configurations are negligible if $s_N \ll \sqrt{N}$. We recall here that we use the notation $a_N \ll b_N$ when the ratio a_N/b_N goes to zero as $N \rightarrow \infty$.

To consider greater scales that we study in this paper (up to $N^{1/2+\eta}, \eta < 1/22$), we need to improve an upper bound at the l.h.s. of (12). Consider a closed even path D_j . Without loss of generality, we can assume $j = 1$, and consider the path D_1 . Let us denote by $x_1(t)$ the simple random walk trajectory trajectory associated with D_1 and by $2s'_1$, the length of D_1 .

Assume also that D_1 has been glued from $I'_1 + 1 \geq 2$ paths (without loss of generality, we can assume that these paths (in the order of gluing) are $W_1, W_2, \dots, W_{I'_1+1}$). Let us denote by t_1 the moment of time in the path D_1 that corresponds to the instant when we glued W_1 and W_2 together to form $W_1 \vee W_2$, let us denote by $t_2 > t_1$ the moment of time that corresponds to the instant when we glued $W_1 \vee W_2$ with W_3 to form $W_1 \vee W_2 \vee W_3$, and so on. Finally, we denote by $t_{I'_1} > t_{I'_1-1}$ the moment of time that corresponds to the instant of switch when we glued $W_1 \vee W_2 \dots \vee W_{I'_1}$ and $W_{I'_1+1}$ to form $W_1 \vee W_2 \dots \vee W_{I'_1+1} = D_1$. Let us

denote by l_j the length of the path W_j , $1 \leq j \leq I'_1 + 1$. It follows from the gluing procedure that the random walk trajectory $x_1(t)$ does not descend below the level $x_1(t_1)$ during $[t_1, t_1 + l_2 - 1]$. Also, once l_2 is given, there are at most l_2 possible choices for the origin of the path W_2 when we reconstruct it from D_1 . When we glue the path W_3 to $W_1 \vee W_2$ in such a way that the edge along which we glue them belongs to W_1 then $t_2 \geq t_1 + l_2$, and the random walk trajectory $x(t)$ does not descend below the level $x_1(t_2)$ during the interval $[t_2, t_2 + L_3]$, $L_3 = l_3 - 1$. We also remark that there are at most l_3 possible choices for the origin of the path W_3 . If instead the edge along which we glue W_3 to $W_1 \vee W_2$ belongs to W_2 , then we have $t_2 \in (t_1, t_1 + l_2)$, and the random walk trajectory does not descend below the level $x_1(t_2)$ during the interval $[t_2, t_2 + L_3]$, $L_3 = l_2 + l_3 - 2$. Again, there are at most l_3 possible choices for the origin of the path W_3 . A similar reasoning can be applied when we consider the gluings of W_4 to $W_1 \vee W_2 \vee W_3$, and so on.

If $I'_1 = 1$, i.e. D_1 was obtained by gluing just two paths W_1 and W_2 , we see that the number of preimages of D_1 is bounded from above by

$$\sum_{t_1 \leq 2s'_1} \sum_{l_2 \leq 2s'_1 - t_1} 1_{\{x_1(t) \geq x_1(t_1), t \in [t_1, t_1 + l_2]\}} 2l_2 \leq (4s'_1)K_N(x_i(\cdot)), \tag{13}$$

where

$$K_N(x_i(\cdot)) = \sum_{t_1 \leq 2s'_1} \sum_{l_2 \leq 2s'_1 - t_1} 1_{\{x_1(t) \geq x_1(t_1), t \in [t_1, t_1 + l_2]\}}. \tag{14}$$

We note that the factor $2l_2$ in (13) comes from the determination of the origin and the direction of W_2 , and the bound $2l_2 \leq 4s'_1$ is trivial.

In the general case $I'_1 \geq 1$, the number of preimages of D_1 is bounded from above by

$$\sum_{0 < t_1 < t_2 < \dots < t_{I'_1} < 2s'_1} \prod_{j=1}^{I'_1} \left(\sum_{L_{j+1} \leq 2s'_1 - t_j} 1_{\{x_1(t) \geq x_1(t_j), t \in [t_j, t_j + L_{j+1}]\}} 2l_{j+1} \right), \tag{15}$$

where, as we explained above, L_j is a sum of $l_j - 1$ and some of the $(l_i - 1)$ with indices $i < j$. Bounding $\prod_j 2l_{j+1}$ from above by $2^{I'_1} ((2s'_1 + 2I'_1)/I'_1)^{I'_1} \leq \text{Const}^{I'_1} (2s'_1)^{I'_1}$, we obtain that in the general case $I'_1 \geq 1$, the number of preimages of D_1 is bounded from above by

$$\text{Const}^{I'_1} \binom{2s'_1}{I'_1} K_N^{\otimes I'_1}(x_1(\cdot)), \tag{16}$$

where

$$K_N^{\otimes I'_1}(x_1(\cdot)) = \sum_{0 < t_1 < t_2 < \dots < t_{I'_1} < 2s'_1} \prod_{j=1}^{I'_1} \left(\sum_{L_{j+1} \leq 2s'_1 - t_j} 1_{\{x_1(t) \geq x_1(t_j), t \in [t_j, t_j + L_{j+1}]\}} \right), \tag{17}$$

Since the matrix entries of A_N are of order of $1/\sqrt{N}$, the “restoration” of each of I'_1 edges during the reconstruction of the paths W_i ’s from D_1 contributes the additional factor $(\text{const}/N)^{I'_1}$. Therefore, we need to bound from above the number of preimages of D_1 times the factor $(\text{const}/N)^{I'_1}$. Let us denote by $\mathbb{E}_{2s'_1}$ the expectation with respect to the uniform distribution on the set of Dyck paths of length $2s'_1$. We are looking for an upper estimate on

$$(\text{const}/N)^{I'_1} \text{Const}^{I'_1} \binom{2s'_1}{I'_1} \mathbb{E}_{2s'_1} (K_N^{\otimes I'_1}(x_1(\cdot))).$$

The calculation of the upper bound are similar to the ones in Lemma 1 of Ref. 10 (see also the discussion on page 128 of Ref. 11). For example, it was shown in Ref. 10 that

$$\mathbb{E}_{2s} \left(\sum_{t_1 \leq s} 1_{\{x(t) \geq x(t_1), t \in [t_1, t_1 + s]\}} \right) = 2\sqrt{\frac{s}{\pi}}(1 + o(1)). \tag{18}$$

Almost identical calculations establish that

$$\mathbb{E}_{2s'_1} (K_N(x_1(\cdot))) \leq \text{Const}(2s'_1)^{3/2} \tag{19}$$

and, in general,

$$\mathbb{E}_{2s'_1} (K_N^{\otimes I'_1}(x_1(\cdot))) \leq (\text{Const}(2s'_1)^{3/2})^{I'_1} \tag{20}$$

for some constant $\text{Const} > 0$. For the convenience of the reader, we sketch the proof of (19) and (20) in the Appendix.

As a result, we obtain

$$\begin{aligned} (\text{const}/N)^{I'_1} \text{Const}^{I'_1} \binom{2s'_1}{I'_1} \mathbb{E}_{2s'_1} (K_N^{\otimes I'_1}(x_1(\cdot))) &\leq \binom{2s'_1}{I'_1} \\ &\times \left(\frac{\text{Const} \times s^{3/2}}{N} \right)^{I'_1} \ll \binom{2s'_1}{I'_1} \end{aligned} \tag{21}$$

as long as $s_N \ll N^{2/3}$. Again, this estimate is enough for our purposes to show in Sec. 3.1.2 (see (27), (28)) that the contribution of such configurations is negligible in the large- N -limit.

3. THE INSERTION PROCEDURE AND THE CASE WHERE $s_N \ll \sqrt{N}$

In this section, we prove the following result. Denote by Z_e (resp. Z_o) the contribution of even (resp. odd) paths.

Proposition 3.1. *Let s_N be some sequence such that $s_N \rightarrow \infty$, $s_N \ll \sqrt{N}$ as $N \rightarrow \infty$. Then*

$$\mathbb{E}[Tr A_N^{2s_N}] = Z_e(1 + o(1)) = (1 + o(1))NT_{0,2s_N}\sigma^{2s_N}. \tag{22}$$

In view of the result of Ref. 10, Proposition 3.1 is a special case of Proposition 1.1 (in the regime $s_N \ll \sqrt{N}$). The proof of Proposition 3.1 is the goal of the whole section. We first define the basic combinatorial tool, namely the *insertion procedure* that we will use to estimate the expectation (2). The basic idea is the following. The contribution of even paths is well-known from the calculations presented in Ref. 10. We then estimate the number of ways to insert non-returned edges in an even path in such a way that the final path has a given number of odd edges (each being read at least three times). In the process, we estimate the weight of the final path in terms of the weight of the initial even path. This finally allows us to consider the contribution of odd paths to the expectation (2).

3.1. The Insertion Procedure

We are going to define the procedure which is the reverse one to the gluing procedure described in Sec. 2. The new procedure will prescribe how to insert sequences of odd edges into a given path \mathcal{P}' to construct the path $\mathcal{P} = \{(i_0, i_1), (i_1, i_2), \dots, (i_{s_N-1}, i_{s_N})\}$. This reverse procedure will allow us to estimate the contribution of odd paths. To this aim, we consider all possible paths \mathcal{P}' and all possible ways to insert odd edges into such paths. In the gluing procedure, when some of the J vertices are repeated, there are multiple ways to glue the paths \mathcal{P}_i . The counterpart for the insertion procedure will be that, given a path \mathcal{P}' and a sequence of J instants along this path, each time a vertex occurs $2i$ times as an endpoint of a sequence of odd edges, the insertion procedure will be non determined.

3.1.1. The Simple Case: Case A

Assume given a closed even path \mathcal{P}' , of the length $2m = 2s_N - 2l$. Here we assume that we know all the edges read in \mathcal{P}' and the order in which these edges are read. To reconstruct the path \mathcal{P} we need to construct the subpaths $\mathcal{P}_i, i = 0, \dots, J$ from the path \mathcal{P}' , and insert between the \mathcal{P}'_i s the J sequences S_1, \dots, S_J of odd edges. To this end, we first choose J vertices amongst the vertices \mathcal{P}' . There are at

most $\binom{2m}{J}$ such choices. The chosen J vertices then split \mathcal{P}' into $J + 1$ subpaths $\mathcal{R}_i, i = 0, \dots, J$, so that these vertices together with the starting point of the path \mathcal{P}' are the endpoints of the subpaths $\mathcal{R}_i, i = 0, \dots, J$. We also set $\mathcal{P}_0 = \mathcal{R}_0$. The subpaths $\mathcal{P}_i, i = 1, \dots, J$ differ from $\mathcal{R}_i, i = 1, \dots, J$ only by the order in which they are read and (perhaps) the directions in which they are read. Since there are 2 choices for the direction of each of the paths and $J!$ ways in which one can order the paths, there are at most $J!2^J$ ways to reconstruct $\mathcal{P}_i, i = 1, \dots, J$ from $\mathcal{R}_i, i = 1, \dots, J$. We can choose the number of unreturned edges we assign to each of the sequences $S_i, 1 \leq i \leq J$ in $\binom{2l}{J}$ ways (indeed, we look for the number of ways to write $2l$ as a sum of J positive integers). Finally, we choose an ordered collection of $2l - J$ edges from the set of edges of \mathcal{P}' . We can do it in at most $\frac{(2m)!}{(2m-2l+J)!}$ ways. It should be noted that it is enough to select $2l - J$ and not $2l$ odd edges since we already know the end points of each sequence S_i of odd edges.

Multiplying these factors together, we obtain

$$\binom{2m}{J} J! 2^J \binom{2l}{J} \frac{(2m)!}{(2m - 2l + J)!}.$$

The last thing that we have to take into account is that the weight $\mathbb{E}\left(\prod_{j=0}^{2s_N-1} \frac{a_{ij_{j+1}}}{\sqrt{N}}\right)$ of the path \mathcal{P} is different from that of \mathcal{P}' since the odd edges from the path \mathcal{P} appear one less time in the path \mathcal{P}' . As the marginal distribution of the matrix entries a_{ij} has bounded support, it follows that the weight of the path \mathcal{P} is at most $(K/\sqrt{N})^{2l}$ times the weight of the path of \mathcal{P}' , where K is some constant that depends only on the marginal distribution of the matrix entries. On the other hand, the following upper bound for the total weight of even paths of length $2s - 2l$ can be inferred from Ref. 11. Define

$$Z(l) := \sum_{\text{even paths } \mathcal{P}'} \mathbb{E} \left[\prod_{j=0}^{2s-2l-1} \frac{a_{ij_{j+1}}}{\sqrt{N}} \right].$$

Lemma 3.1. *There is a constant C_1 , independent of l , such that for any sequence $s_N \ll N^{2/3}$,*

$$Z(l) \leq C_1 N T_{0,2s_N-2l} \sigma^{2s_N-2l}, \quad \text{where} \quad T_{0,2s_N-2l} = \frac{(2s_N - 2l)!}{(s_N - l + 1)!(s_N - l)!}.$$

From Lemma 3.1 and the above estimate on the number of preimages of paths \mathcal{P}' , we deduce that the contribution of paths \mathcal{P} such that $l > 0$ and $I = 1$ is at

most

$$\begin{aligned}
 & \sum_{l=1}^{s_N-1} C_1 N \frac{(2s_N - 2l)!}{(s_N - l)!(s_N - l + 1)!} \frac{\sigma^{2s_N-2l}}{N^l} \sum_{J=1}^{2l} \binom{2s_N - 2l}{J} J! 2^J \binom{2l}{J} \\
 & \quad \times \frac{(2s_N - 2l)!}{(2s_N - 4l + J)!} \frac{K^{2l}}{N^l} \\
 & \leq \sum_{l=1}^{s_N-1} C_1 N \frac{(2s_N - 2l)!}{(s_N - l)!(s_N - l + 1)!} \sigma^{2s_N-2l} \left(\frac{16K(s_N - l)}{\sqrt{N}} \right)^{2l}. \tag{23}
 \end{aligned}$$

In the case where $s_N \ll \sqrt{N}$, this is enough to show that the contribution of paths with odd edges is negligible in the large N limit compared to the r.h.s. of (22).

3.1.2. The Cases B and C

We start with Case B. Assume that the closed even paths corresponding to each of the $I \geq 2$ clusters have respective lengths $2s_i, i = 0, \dots, I - 1$ where $\forall i, s_i > 0$ and $\sum_{i=0}^{I-1} 2s_i = 2s - 2l$. We denote these closed even paths by $W_i, i = 0, \dots, I - 1$ as in Sec. 2.1. Let us first assume that we know the first path W_0 completely, in other words, we know its starting point, all edges read in W_0 , and the order in which these edges are read. Since W_0 is a closed even path, its contribution to $\mathbb{E}[Tr A_N^{2s_N}]$ was studied completely in Ref. 11 and can be written as $N T_{0,2s_0} \sigma^{2s_0} (1 + o(1))$. We recall that the factor N up front appears because we have N choices for the starting point of W_0 . As noted in Sec. 2.1 and the beginning of Sec. 2.2, the starting points of each of the last $I - 1$ paths W_1, \dots, W_{I-1} are marked vertices of \mathcal{P}' . Therefore, provided we know the set of all marked vertices of \mathcal{P}' , we can choose the origins of W_1, \dots, W_{I-1} in at most $\binom{2s_N - 2l}{I-1}$ ways. We recall (see also Refs. 10–12) that we select the set of marked edges at the very beginning of the counting procedure. The order in which we choose the origins is irrelevant since, in view of the *insertion procedure* defined above, it is the unordered collection of the $I - 1$ paths which is relevant for the computation here, once the first path is chosen. In addition to the chosen $I - 1$ origins, we also choose $J - (I - 1)$ vertices amongst the vertices of \mathcal{P}' in at most $\binom{2s_N - 2l - I + 1}{J - I + 1}$ ways. This gives us J endpoints of the sequences of odd edges S_1, S_2, \dots, S_J described at the beginning of Sec. 2.1. As in the previous subsection, the choice of these J vertices splits \mathcal{P}' into $J + 1$ subpaths $\mathcal{R}_i, i = 0, \dots, J$. Again, we set $\mathcal{P}_0 = \mathcal{R}_0$, and note that the subpaths $\mathcal{P}_i, 1 \leq i \leq J$ differ from the subpaths $\mathcal{R}_i, 1 \leq i \leq J$ only by their ordering and their directions. Therefore, there are at most $2^J J!$ ways to reconstruct $\mathcal{P}_i, 1 \leq i \leq J$ from $\mathcal{R}_i, 1 \leq i \leq J$. Following the same calculations as in Case A, we arrive at the following upper bound on the

contribution of paths \mathcal{P} from Case B:

$$\sum_{l=1}^{s-1} \sum_{j=1}^{2l} 2^j J! \binom{2l}{j} \frac{(2s_N - 2l)!}{(2s_N - 4l + J)!} K^{2l} N^{-l} \sum_{I=2}^J \binom{2s_N - 2l}{I - 1} \times \binom{2s_N - 2l - I + 1}{J - I + 1} \sum_{s_0, \dots, s_{I-1}; \sum_i s_i = 2s - 2l} N \prod_{i=0}^{I-1} C_1 T_{0, 2s_i} \sigma^{2s_i}. \quad (24)$$

It can indeed be inferred from computations as in Ref. 11 that typical clusters of paths W_0, \dots, W_{I-1} do not share edges, which would be edges read at least four times in \mathcal{P}' . To simplify the last formula, we note that

$$\binom{2s_N - 2l}{I - 1} \binom{2s_N - 2l - I + 1}{J - I + 1} \leq 2^J \binom{2s_N - 2l}{J}, \quad (25)$$

and observe that

$$\sum_{I=2}^J \sum_{s_0, \dots, s_{I-1}; \sum_i s_i = 2s - 2l} \prod_{i=0}^{I-1} T_{0, 2s_i} \leq \text{const}^J T_{0, 2s - 2l} \leq \text{const}^{2l} T_{0, 2s - 2l}, \quad (26)$$

where $\text{const} > 0$ is some constant which essentially follows from the inequality

$$\sum_{k=1}^{s-1} \frac{1}{k^{3/2}} \frac{1}{(s - k)^{3/2}} \leq \text{const} s^{3/2},$$

for some appropriate $\text{const} > 0$. It follows from (25) and (26) that the upper bound in (24) is negligible compared to the contribution given by the closed even paths (i.e. $l = 0$) to $\mathbb{E}[Tr A_N^{2s_N}]$. Now we turn our attention to Case C. In other words, we assume that at least one of the paths W_0, \dots, W_{I-1} has an odd edge. As we explained in the beginning of Sec. 2.2, the counting in this case can be reduced to Case B or Case A. Namely, we employ the second gluing procedure to construct $I - I_1$ closed even paths D_0, \dots, D_{I-I_1-1} from the paths W'_i 's. Here we consider the case where $I - I_1 > 1$ (thus reducing Case C to Case B). If $I - I_1 = 1$, one reduces Case C to Case A by similar arguments. Let us assume that D_i was obtained by gluing together $I'_i + 1$ paths, where $I'_i \geq 0, 0 \leq i \leq I - I_1 - 1$. As we have shown in the formulas (12) and (21) derived in Sec. 2.2, when we reconstruct D_i from the corresponding subset of paths from $\{W_0, \dots, W_{I-1}\}$, we obtain a factor $\left(\frac{(s'_i)^{3/2}}{N}\right)^{I'_i} \binom{2s'_i - 2l}{I'_i}$. Since

$$\binom{2s_N - 2l}{I - I_1 - 1} \sum_{I'_0, I'_1, \dots} \prod_{i=0}^{I-I_1-1} \left(\frac{(s'_i)^{3/2}}{N}\right)^{I'_i} \binom{2s'_i}{I'_i} \leq \left(\frac{s^{3/2}}{N}\right)^{I_1} \binom{2s - 2l}{I - 1} \text{Const}^l, \quad (27)$$

we can continue the calculations along the same lines as in Case B, just replacing the factor $\binom{2s_N-2l}{I-1}$ in (24), (25) by the l.h.s. of (27) and summing over $1 \leq I_1 < I$. In other words, one can estimate the upper bound

$$\begin{aligned} & \sum_{l=1}^{s-1} \sum_{j=1}^{2l} 2^j J! \binom{2l}{J} \frac{(2s_N - 2l)!}{(2s_N - 4l + J)!} K^{2l} N^{-l} \sum_{I=2}^J \binom{2s_N - 2l - I + 1}{J - I + 1} \\ & \times \sum_{I_1=1}^{I-1} \binom{2s_N - 2l}{I - I_1 - 1} \sum_{\sum_i s'_i = 2s_N - 2l - 2I_1} \sum_{i=0}^{I-I_1-1} N \prod_{i=0}^{I-I_1-1} C_1 T_{0,2s'_i} \sigma^{2s'_i} \left(\frac{s^{3/2}}{N}\right)^{I'_i} \binom{2s'_i}{I'_i} \end{aligned} \tag{28}$$

using (27), (25), and (26). In particular, the expression in (28) is negligible compared with the upper bound (24) from Case B for $s_N^{3/2} \ll N$.

4. GREATER SCALES

In this Section, we set $s_N = N^{1/2+\eta}$ where $\eta < 1/22$. We prove the following result. Let Z_e (resp. Z_o) be as before the contribution of even (resp. odd) paths.

Proposition 4.1. *Assume that $s_N = N^{1/2+\eta}$ where $\eta < 1/22$. Then, one has that*

$$\mathbb{E}[\text{Tr } A_N^{s_N}] = Z_e(1 + o(1)).$$

To prove Proposition 4.1, we refine the procedure we have used for powers $s = s_N \ll \sqrt{N}$ in the previous Section. In particular, one has to refine the numbering of the preimages of a given path \mathcal{P}' . As before, we consider separately the cases where $I = 1$ (Case A) and $I > 1$.

4.1. The Case Where $I = 1$ (Case A)

4.1.1. Obtaining a Bound on l

The aim of the arguments presented here is to show that the contribution of paths with large l is negligible. We first establish a Proposition which refines the bound on the number of ways to insert the odd edges.

Proposition 4.2. *There exists a constant $C > 0$ such that the number of possible ways to choose and insert the odd edges is at most*

$$\sum_{1 \leq J \leq 2l} \sum_{1 \leq c \leq J} \frac{1}{c!} s_N^c s_N^l \frac{1}{(J-c)!} s_N^{J-c} C^{2l}, \tag{29}$$

Proof of Proposition 4.2: We start with a few remarks on how the odd edges are split into cycles. Consider a path \mathcal{P} with $2l$ odd edges split into J sequences S_1, \dots, S_J as described in Sec. 2.1. One can reformulate Lemma 2.1, as a statement that the set of the odd edges can be viewed as a union of cycles. Note that the number of cycles c a priori is not well defined if the cycles in the union are not disjoint (in other words, if there is a vertex v which is an end point of more than two odd edges). To make the definition of c precise, we have to show how we construct the cycles. Recall the *gluing procedure* described in Subsection 2.1. Each time we glue two subpaths at a common vertex v during the *gluing procedure*, we shall do the following. We shall add the two corresponding sequences of odd edges, chosen from the set of sequences $\{S_1, \dots, S_J\}$ so that both of the sequences have v as an endpoint, to the cycle, or we shall start a new cycle by attaching these two sequences together. Following the gluing procedure to the end, we end up with a set of c cycles of odd edges.

A useful observation is that if one can insert c cycles of odd edges in a given path \mathcal{P}' , then \mathcal{P}' has at least c self-intersections. This can be seen as follows. Along each cycle of odd edges, we “orient” the odd edges according to the direction they are read for the first time in \mathcal{P}' . Due to the cycle structure, one of the two things happens : either (a) there are two edges that point to the same vertex, implying that this vertex is necessarily a vertex of self-intersection in a sense of Refs. 10, 11, or (b) all edges in the cycle have the same “orientation” in which case the starting point of the cycle is a point of self-intersection.

Now, we refine our insertion procedure using the cycle structure. Assume that a path \mathcal{P}' is given. Let $1 \leq c \leq J$ be the number of cycles to be inserted. To insert the $2l$ odd edges, we apply the following *insertion procedure*:

1. We choose the instants t_1, t_2, \dots, t_c along \mathcal{P}' where the c cycles start. One can do it in $\binom{2s_N}{c}$ ways. This defines c vertices which are not necessarily distinct. The smallest t_i determines the first cycle.
2. We choose the number of odd edges that will belong to each of the cycles. The number of ways to write $2l$ as a sum of c positive integers is at most $\binom{2l}{c} \leq 2^{2l}$.
3. We choose the $2l$ odd edges. For this, one can note that it is enough to choose every other edge inside each of the cycles. For instance, if there are c_1 odd edges in cycle 1, it is enough to choose $c_1/2$ edges if c_1 is even and $(c_1 - 1)/2$ if c_1 is odd (since we have already chosen the starting points of the cycles). Note that this also defines, if there is an ambiguity, the cycle to which each edge belongs. It follows that at step 3, we can choose the odd edges in at most $(2s_N)^l$ ways (later we will refine this bound a little).
4. We choose the $J - c$ moments in the cycles (in addition to the c moments that are the starting points of the cycles). This choice will give us the set of vertices that appear as the endpoints of the sequences of odd edges S'_i .

For this it is enough to choose $J - c$ edges (out of $2l$ odd edges) starting or ending a sequence of odd edges and decide for each of the chosen $J - c$ edges whether it starts or finishes a sequence. There are at most $2^{J-c} \binom{2l}{J-c} \leq 2^{4l}$ such possible choices.

At this point, we are given cycles where all the edges are known and where we also know the end points of all $S_i, 1 \leq i \leq J$. There remains to plug in the J sequences of odd edges into the path \mathcal{P}' . The easiest case is when the J vertices occurring at the J endpoints of the J sequences are pairwise disjoint. We note that we are talking about J and not $2J$ endpoints of the sequences $S_i, 1 \leq i \leq J$ since these sequences are the segments of the cycles. We have already chosen the c instants t_1, t_2, \dots, t_c where the cycles start. Therefore, it is enough to choose a subset of $J - c$ instants in \mathcal{P}' . Indeed, suppose that we have just chosen such a subset of $J - c$ instants in \mathcal{P}' . In addition to the c chosen instants in \mathcal{P}' corresponding to the starting points of the c cycles, this gives us the J instants in \mathcal{P}' . To form the path \mathcal{P} from \mathcal{P}' , one first copies the edges of \mathcal{P}' until one meets the vertex v_1 that starts the first cycle. Then we plug in the sequence of odd edges that starts at v_1 . Let us call by w_1 the other endpoint of this sequence. Having inserted this first sequence, we need to know two things to proceed. First, we need to know the corresponding instant in \mathcal{P}' where w_1 occurs. In the case when the J vertices occurring at the J endpoints of the J sequences are pairwise disjoint, we have at most one choice for this instant among the J instants chosen above in \mathcal{P}' . We then proceed by reading a portion of the path \mathcal{P}' starting from w_1 . To do this, we need to decide in which direction to read a portion of \mathcal{P}' . Namely, we have to decide whether to read the portion of \mathcal{P}' on the right or on the left of w_1 , or equivalently whether we will go from w_1 to the right (in the direction of \mathcal{P}') or to the left (reversing the direction of the corresponding edges in \mathcal{P}'). Once we decided on this, we read the edges of \mathcal{P}' until we meet the next vertex from the set of the J selected instants. At this vertex, we plug in the next sequence of odd edges (in general, we will have to choose one of the two possible directions), and we iterate the procedure.

Iterating the procedure, we will have to choose a direction at most $4l$ times, which gives us a factor 2^{4l} . Note that the procedure also defines the order in which the cycles are met in \mathcal{P} . At this point, under the assumption that the J vertices at the endpoints of the J sequences of odd edges are pairwise disjoint, the total number of ways to choose and insert the odd edges is at most

$$2^{11l} s_N^c s_N^l \frac{1}{c!(J-c)!} s_N^{J-c}. \tag{30}$$

Let us now consider the case when the J vertices occurring as the endpoints of the sequences of odd edges have been chosen and are not pairwise disjoint. Suppose for example that the vertex w_1 occurs $A(w_1)$ times as an endpoint of the

$A(w_1)$ sequences of odd edges. If we try to implement the strategy outlined above, once we have inserted the first sequence of odd edges, there will be at most $A(w_1)!$ possibilities for the choice of the corresponding instance in \mathcal{P}' where w_1 occurs. The same argument holds for the other “multiple” vertices as well. Therefore, the total number of ways to choose and insert the odd edges is at most

$$2^{11l} s_N^c s_N^l \frac{1}{c!(J-c)!} s_N^{J-c} \prod_{v \text{ multiple}} A(v)! \leq 2^{13l} s_N^c s_N^l \frac{1}{J!} s_N^{J-c} \prod_{v \text{ multiple}} A(v)!, \tag{31}$$

where we estimated $\frac{1}{c!(J-c)!}$ from above by $\frac{2^J}{J!} \leq \frac{2^{2l}}{J!}$. While the factor $\prod_{v \text{ multiple}} A(v)!$ in (31) can be quite large in the case of “multiple” vertices, the path \mathcal{P} can be glued from the subpaths \mathcal{P}_i in many different ways (see (11)) which cancels this factor once we take into account the overcounting. Namely, suppose the path \mathcal{P} has a vertex v occurring A times as an endpoint of a sequence of odd edges. For such a path \mathcal{P} , it follows that there are roughly speaking $A!$ ways to glue the subpaths \mathcal{P}_i in the process of constructing \mathcal{P}' . Denote by E'_i the number of vertices v amongst the $J - c$ endpoints of the sequences of odd edges for which $A(v) = 2i$. Then

$$\sum_{i=1}^{J-c} i E'_i = J - c.$$

Note that E'_i is uniquely determined by the choice of the unreturned edges and the choice of the J instants. Combining (30), (31) and (11), the total number of ways to choose and insert the odd edges, divided by the number of possible gluings of the corresponding paths \mathcal{P}_i is at most (given c and J)

$$s_N^c s_N^l \frac{1}{J!} s_N^{J-c} C^{2l}, \tag{32}$$

where C is a sufficiently large constant. This holds whether the J vertices are distinct or not. Now, one has that

$$\sum_{\mathcal{P} \text{ with } 2l \text{ unreturned edges}} \mathbb{E}(\mathcal{P}) \leq K^{2l} \sum_{I \geq 1} \sum_J \sum_{E_1, \dots, E_J} \prod_{i=2}^J \frac{1}{(i!)^{E_i}} \sum_{\mathcal{P}'} N(2l, \mathcal{P}' | E_i) \mathbb{E}[\mathcal{P}'],$$

where $N(2l, \mathcal{P}' | E_i)$ denotes the number of possible choices and insertions of the $2l$ edges into a path \mathcal{P}' , knowing that amongst the J endpoints the E_i ones occur $2i$ times. This follows from the fact that the insertion procedure is the reverse one to the gluing procedure. Thus, it is enough to consider the number of possible choices and insertions of odd edges divided by the number of possible gluings of the image path (for any $I \geq 1$) to estimate the contribution of paths with unreturned edges. It finishes the proof of Proposition 4.2. \square

Thanks to Proposition 4.2, one can first show that paths with many odd edges are negligible. We obtain the following bound.

Proposition 4.3. *Assume that $\eta < 1/22$ and let $\epsilon < 1/4 - 5\eta/2$. Then the paths with more than $N^{1/4+\eta/2-\epsilon/2}$ odd edges yield a negligible contribution.*

Proof of Proposition 4.3: We start with a few remarks:

- It is easy to show that the contribution from the paths for which $l \geq \text{Const}N^{1/4+3\eta/2}$ is negligible, provided Const is large enough, by using the Stirling’s formula and Proposition 4.2.
- It is also clear from Proposition 4.2 that the paths for which $J < l$ and $l > N^{2\eta}$ yield a negligible contribution. We note that $1/4 + \eta/2 - \epsilon/2 \geq 2\eta$ for $\eta < 1/22$ and $\epsilon < 1/4 - 5\eta/2$. Thus from now on, we consider paths such that $J \geq l$ and $l \leq \text{Const}N^{1/4+3\eta/2}$.

In what follows, we first restrict our attention to the case $I = 1$ (i.e. when \mathcal{P}' is just one closed even path). As in Sec. 3, the case $I > 1$ follows in a rather straightforward fashion from the case $I = 1$ (this will be done in Sec. 4.2.) For the rest of the proof we essentially need to show that $l \ll \sqrt{s_N} = N^{1/4+\eta/2}$ in the paths that give the main contribution. We need to refine our estimates. When choosing the $2l$ edges occuring in the c cycles, we have already seen that it is enough to choose every other edge. Therefore, once we know the origin v_0 of a cycle, we then choose not the very first edge of the cycle for which v_0 is a left end point but rather the next edge. The number of ways to choose this edge (the second one among the edges of the cycle) is at most

$$\mathbf{M}(\mathbf{v}_0) := \sum_{v_1:(v_0v_1) \in \mathcal{P}'} v_N(v_1), \tag{33}$$

where $v_N(v_1)$ is the number of edges which have v_1 as an end point. The quantity (33) is an upper bound of the number of vertices which are at a distance 2 from a given vertex. Here the denomination v_2 is at a distance 2 from a vertex v means that there exists a vertex v_1 such that (vv_1) and (v_1v_2) are non-oriented edges of \mathcal{P}' . □

Assume first that $\max_{v \in \mathcal{P}'} \sum_{v_1:(vv_1) \in \mathcal{P}'} v_N(v_1) \leq N^{1/2-\eta-\epsilon}$ for all $0 < \epsilon < 1/4 - 5\eta/2$. Then the number of ways to choose and insert the unreturned edges is at most of order

$$\frac{1}{J!} s_N^J N^{-l} (N^{1/2-\eta-\epsilon})^l \leq \frac{1}{2l!} (N^{1/2+\eta-\epsilon})^l,$$

so that the contribution of paths for which $l \geq \text{Const}N^{1/4+\eta/2-\epsilon/2}$ (where Const is sufficiently large) is negligible for all $0 < \epsilon < 1/4 - 5\eta/2$. which proves the statement of Proposition 4.3 in this case.

Now let us assume that $\max_{v \in \mathcal{P}'} \sum_{v_1: (vv_1) \in \mathcal{P}'} v_N(v_1) > N^{1/2-\eta-\epsilon}$ for some fixed $0 < \epsilon < 1/4 - 5\eta/2$. This means that there are at least $N^{1/2-\eta-\epsilon}$ vertices at a distance 2 from some vertex v in \mathcal{P}' . Our goal is then to show that the paths for which $M(v) > N^{1/2-\eta-\epsilon}$ for some vertex v are negligible. To do this, we first need to introduce the following quantity. Denote by κ the number of self-intersections of type greater than 2 plus the number of non-closed vertices in \mathcal{P}' . For the definitions of the self-intersections and non-closed edges we refer the reader to Refs. 11, 12. In the notations of Refs. 11, 12, we have $\kappa = r + \sum_{k>2} kn_k$, where r is the number of non-closed vertices and n_k is the number of the k -fold self-intersections. As $l \leq \text{Const}N^{1/4+3\eta/2}$, one has that κ is at most of the order $O(N^{1/4+3\eta/2})$ in the typical paths as well. The reason is as follows. Below, we appeal to some computations made in Ref. 11. Let $\max x(t)$ be the maximum level reached by the trajectory of a Dyck path of length $2s - 2l$. Let also \mathbb{P}_s denote the uniform distribution on the set of Dyck paths of length $2s$. It can be shown that there exist constants independent of l such that

$$\mathbb{P}_{s-l}(\max x(t) = k) \leq C_1 \exp\{-C_2 k^2/(s - l)\}. \tag{34}$$

Furthermore, the contribution of paths with $2s - 2l$ edges can be estimated from above by (see e.g. Ref. 11)

$$\sigma^{2s-2l} T_{0,2s-2l} e^{N^{2\eta}} \mathbb{E}_{s-l} \left[\sum_{r, n_k, k \geq 3} \frac{1}{r!} \left(\frac{s_N \max x(t)}{N} \right)^r \prod_{k \geq 3} \frac{1}{n_k!} \left(\frac{C s_N^k}{N^{k-1}} \right)^{n_k} \right]. \tag{35}$$

The sum in (35) is over the number of non-closed edges $r \geq 0$ and the numbers n_k of the self-intersections of order $k \geq 3$. The factor $e^{N^{2\eta}}$ in (35) is a rough upper bound of $\frac{1}{(n_2-r)!} \left(\frac{s_N^2}{N} \right)^{n_2-r}$. It follows from (32) that the insertion of odd edges multiplies the contribution of a path of $T_{0,2s-2l}$ by a factor of order at most $\frac{1}{(2l)!} (s_N^3/N)^l \leq \text{Const}N^{1/4+3\eta/2}$. As a result, one can see that

- In typical paths, independently of l , the maximal level reached by a trajectory is not greater than $B_2 \sqrt{s_N} \sqrt{N^{1/4+3\eta/2}}$ which is of order $B'_2 N^{3/8+5\eta/4}$.
- There are no vertices of type greater than $C N^{1/4+3\eta/2} / \ln N$ in typical paths.

Let then set $\kappa_{o,1} = \sum_{i=1}^{200} N_i$ and $\kappa_{o,2} = \sum_{i>200} i N_i$. Using the above calculations, one can deduce that the contribution of even paths for fixed $r, \kappa_{o,1}, \kappa_{o,2}$ is at most of order

$$\sigma^{2s-2l} T_{0,2s-2l} e^{N^{2\eta}} \frac{1}{r!} (N^{-1/8+9\eta/4})^r \frac{1}{\kappa_{o,1}!} (N^{3\eta-1/2})^{\kappa_{o,1}} \left(\frac{C' s_N}{N^{199/200}} \right)^{\kappa_{o,2}}. \tag{36}$$

Now if $\kappa > B_1 N^{1/4+3\eta/2}$ this implies that either

$$r \geq B_1 N^{1/4+3\eta/2}/3 \quad \text{or} \quad \kappa_{o,1} \geq B_1 N^{1/4+3\eta/2}/200 \quad \text{or} \quad \kappa_{o,2} \geq B_1 N^{1/4+3\eta/2}/3.$$

It is easy to see from (36) that one can choose B_1 large enough and η sufficiently small ($\eta < 1/22$ is enough), so that the contribution of odd paths \mathcal{P} obtained from paths for which $\kappa \geq B_1 N^{1/4+3\eta/2}$ is negligible.

It is crucial for the arguments presented below that $1/2 - \eta - \epsilon > 1/4 + 3\eta/2$ since $\epsilon < 1/4 - 5\eta/2$, which implies that $M(v) \gg \kappa, l$ for the paths that give non-negligible contribution. Now, we split $[0, 2s_n]$ into κ intervals, in such a way that inside each of these κ intervals we have no non-closed simple self-intersections and no self-intersections of higher orders. Let us write $M(v) = \sum_{i=1}^{\kappa} m_i(v)$, where $m_i(v)$ is the number of instants (corresponding to the interval number i of the κ intervals into which we just partitioned $[0, 2s_n]$) when one gets within distance 2 from the vertex v . Consider for simplicity the first interval. We first assume that v is not a vertex chosen amongst the κ distinguished vertices. We mark the occurrences of v inside the first interval and denote by $2l_1(v) - 1$ the number of such occurrences. Note that all such moments correspond to the same level of the Dyck trajectory. Thus, calling t_1 (resp. t_2) the first (resp. last) occurrence of v inside the first interval, the sub-trajectory restricted to the interval $[t_1, t_2]$ is the concatenation of l_1 sub-Dyck paths. Consider now the vertices being the endpoints of an up edge which starts at v . We say that such vertices are adjacent to v . In order to have $m_1(v)$ vertices at a distance 2 of v , the trajectory restricted to the first interval must come back a certain amount of times ($m_1(v)$) to the levels of vertices adjacent to v . As v can be a vertex of type 2, one can deduce by using arguments similar to those of Ref. 11 that the probability of this event is at most $l_1^4 \exp(-C_0 m_1(v))$, where C_0 is a positive constant. If v is of type 2, l_1^4 has to be replaced by l_1^8 . Now, when we pass the first vertex of self-intersection at the end of the first interval, it may happen that we come back to the last vertex adjacent to v at some level which is not the same as in the preceding interval. Once the level of this vertex (which can be one of the κ distinguished vertices) is fixed, the picture is the same as in the first interval. Thus crossing one of the κ vertices results in choosing the two moments of time where one comes back to v and to the last adjacent vertex to v . If v is one of the κ distinguished vertices, the picture is essentially the same.

Multiplying the probabilities over $i = 1, \dots, \kappa$ and using the algebraic-geometric inequality, one obtains the upper bound

$$\prod_{i=1}^{\kappa} l_i^4 \exp(-C_0 m_i(v)) \leq \left(\frac{s_N}{\kappa}\right)^{4\kappa} \exp(-C_0 M(v)). \tag{37}$$

It then that the contribution of the paths with $M(v) \geq N^{1/2-\eta-\epsilon}$ can be bounded from above as

$$\frac{1}{2l!} \left(\frac{s_N^3}{N} \right)^l \sum_{M(v) \geq N^{1/2-\eta-\epsilon}} \sum_{\kappa \leq BN^{1/4+3\eta/2}} \left(\frac{s_N^6}{\kappa} \right)^\kappa e^{-C_0 M(v)} \tag{38}$$

and this gives a negligible contribution in the limit $N \rightarrow \infty$. Proposition is proven.

4.1.2. Refining the Number of Insertions

We will now refine our estimate on the number of ways to insert the sequences of odd edges. In order to do this, we need a few definitions.

For $i = 1, \dots, 2l$, let c_i be the number of cycles that consist of i edges. Let also, for any vertex x occuring in the path \mathcal{P}' , denote by $\nu(x)$ the number of distinct edges to which x belongs. Define $\nu_N = \max_{x \in \mathcal{P}'} \nu(x)$. Assume that the c moments of time are chosen when the cycles start.

We first consider the case where $c_1 = 0$ so that $c \leq 2l/3$. Then the following holds:

1. In each cycle of odd length $i = 2i' + 1$, one needs to choose i' edges and the origin of the cycle. In each cycle of even length $i = 2i'$, one needs to choose $i' - 1$ edges, the origin of the cycle, and an edge connected to the origin of the cycle in order to completely define the cycle. From that, we can see that the number of ways to define the cycles is at most

$$s_N^{l - \sum_{i=1}^{2l} c_i} \nu_N^{\sum_{i \text{ even}} c_i} s_N^{\sum_{i \text{ odd}, i \geq 3} c_i / 2}. \tag{39}$$

2. Once the $J - c$ moments of time where we split the cycles are chosen, there are at most ν_N^{J-c} possible choices for the corresponding instants in \mathcal{P}' where the sequences of odd edges will be inserted.

Therefore, the number of ways to choose and insert the cycles, J being given, is at most of order

$$\sum_{c \leq J} \frac{1}{c!} s_N^c s_N^{l - \sum_{i=2}^{2l} c_i} s_N^{\sum_{i \text{ odd}, i \geq 3} c_i / 2} \nu_N^{\sum_{i \text{ even}} c_i} \nu_N^{J-c} \leq s_N^l \nu_N^J \left(\frac{\sqrt{s_N}}{\nu_N} \right)^{\sum_{i \text{ odd}, i \geq 3} c_i}. \tag{40}$$

Let ϵ_N be a sequence going to zero arbitrarily slowly. Assume now that $\nu_N \leq \epsilon_N s_N^\alpha$ where $\alpha = \frac{1}{2} \frac{1/2-2\eta}{1/2+\eta}$. Then one has that

$$\sum_{l \geq 2} \sum_{J=1}^{2l} (s_N^{4l/3} \nu_N^{J-2l/3} N^{-l}) \leq C_2 \sum_{l \geq 2} \epsilon_N^{2l} \leq C_3 \epsilon_N^2.$$

Thus the contribution of the paths for which $\nu_N \leq \epsilon_N s_N^\alpha$ is negligible in the large- N -limit. We denote by $\nu_o := \epsilon_N s_N^\alpha$ this critical scale.

Let now J' be the number of instants chosen amongst the J ones such that the corresponding vertex occurs in more than v_o edges. Denote by $A_i, i = 1, \dots, J'$ the number of times each such vertex occurs as an endpoint of an odd sequence. Recall that $\kappa = \kappa(\mathcal{P}') := r + \sum_{k \geq 3} kn_k$ denotes the number of non-closed vertices of simple self-intersections plus the number of moments of self-intersections of the order three or higher. As $l \leq N^{1/4+\eta/2-\epsilon/2}$ for all $0 < \epsilon < 1/4 - 5\eta/2$, one can easily show that we can restrict our attention to the paths for which $\kappa \leq bN^{1/4+\eta/2-\epsilon/2}$ for some $b > 0$ arbitrarily small. Now $v_o \sim N^{-3\eta/2} \sqrt{s_N} \gg \kappa \sim b\sqrt{s_N}N^{-\epsilon/2}$, as soon as one can choose $\epsilon > 3\eta$. Assuming that $\eta < 1/22$, this clearly holds; thus each time one has more than v_o choices for the moment of insertion, we pay a cost of order

$$s_N^2 \exp \{-v_o/\kappa\} \ll 1,$$

for N large enough.

Then (40) can be refined as follows.

$$\left(\frac{s_N^{4/3}}{N}\right)^l v_o^{4l/3} \prod_{i=1, \dots, J'} \frac{1}{A_i!} \left(\frac{v(x_i)}{v_o}\right)^{A_i} s_N^2 \exp\left\{-\frac{v(x_i)}{\kappa}\right\} \leq \left(\frac{s_N^{4/3}}{N}\right)^l v_o^{4l/3} \varepsilon_N^{J'}. \tag{41}$$

Thus the summation of the above on J, J' , and l yields a negligible contribution as soon as

$$N^{1/4-\eta} \gg N^{1/4+\eta/2-\epsilon/2} \text{ or } \eta < 1/22.$$

We next consider the case where $c_1 > 0$. In this case, a cycle of length one is a loop determined by the moment of time where the loop is started. Then (39) is replaced with

$$\frac{1}{c_1!} s_N^{c_1} \frac{1}{(c - c_1)!} s_N^{c-c_1} s_N^{l-c_1/2-\sum_{i \geq 2} c_i} v_N^{\sum_{i \text{ even}} c_i} s_N^{\sum_{i \text{ odd}, i \geq 3} c_i/2}.$$

Thus (40) becomes

$$\sum_{c_1, c \leq J} \frac{1}{c_1!} s_N^{c_1} \frac{1}{(c - c_1)!} s_N^{c-c_1} s_N^{l-c_1/2-\sum_{i \geq 2} c_i} v_N^{\sum_{i \text{ even}} c_i} s_N^{\sum_{i \text{ odd}, i \geq 3} c_i/2} v_N^{J-c}. \tag{42}$$

And one still has that $\sum_{i \geq 3} c_i \geq (2l - c_1)/3 = \frac{2}{3}(l - c_1/2)$, so that the end of the proof follows.

Remark 4.1. If $c_1 = c \geq 1$, then the path \mathcal{P}' has loops. It can be shown that the contribution of such odd paths is of order $(\frac{s_N^2}{N^{3/2}})^{c_1} N T_{0,2s_N} \sigma^{2s_N}$, which is negligible as $\eta < 1/22$.

4.2. The Case of Multiple Clusters

The computations from the preceding subsection translate to Case B as follows. Assume that I and J are given with $I \leq J$. Assume also given I Dyck paths $Q_i, i \leq I$, such that the total length is $2s_N - 2I$. We first choose the origins of the $I - 1$ last sub-Dyck paths. There are $\binom{2s_N}{I-1}$ possible choices for the set of vertices occuring at the endpoint of clusters. We can indeed assume that the $I - 1$ last sub-Dyck paths are ordered in such a way that their origins u_i satisfy $u_i \leq u_{i+1}$.

Now we choose the set of odd edges and cycles. We also choose respectively the set of J vertices and amongst the latter the set of $I - 1$ vertices. As before, there are

$$\frac{s_N^c}{c!} s_N^I$$

possible such choices. Now there are only $J - c - (I - 1)$ moments of time to be chosen where one inserts sequences of odd edges, since $I - 1$ such moments are determined by the $I - 1$ sub-Dyck paths and the preceding insertions.

Thus the number of ways to choose and insert the sequences of odd edges is at most

$$\begin{aligned} & \sum_{l>1} \sum_{1 \leq J \leq 2l} \sum_{2 \leq I \leq J} \sum_{1 \leq c \leq J} \sum_{s_i > 0: \sum_{i=1}^I s_i = 2s - 2l} T_{0,2s_l} C^l \frac{1}{(I - 1)!(J - c)!} s_N^{c+I-1} s_N^{J-(I-1+c)} \\ & \leq \sum_{l>1} \sum_{2 \leq J \leq 2l} \sum_{1 \leq I \leq J} \sum_{1 \leq c' \leq J} (\text{Const}_1)^l \frac{1}{c'!(J - c')!} s_N^{c'} s_N^{J-c'} \\ & \leq (\text{Const}_2)^l T_{2s-2l} \sum_{l>1} \sum_{2 \leq J \leq 2l} \sum_{1 \leq c' \leq J} (\text{Const}_1)^l \frac{1}{c'!(J - c')!} s_N^{c'} s_N^{J-c'}. \end{aligned} \tag{43}$$

Thus we can use the same analysis as in the preceding case where $I = 1$. We again obtain that the contribution of paths for which $l > 0$ is negligible in the large- N -limit, provided $\eta < 1/22$. The contribution of paths \mathcal{P} falling into Case C can be deduced as before from the analysis of Cases A and B. It is not developed further here. This finishes the proof of Proposition 1.1.

APPENDIX: THE PROOF OF (19) AND (20)

We start with (19). Let us define $r_1 = r_1(t_1) > 0$ as $r_1 = \max\{r : x_1(t) \geq x_1(t_1), t \in [t_1, t_1 + r]\}$. It follows from the definition that $x_1(t_1 + r_1) = x_1(t_1)$ and $x_1(t_1 + r_1 + 1) = x_1(t_1) - 1$. Since $\sum_{l_2 \leq 2s'_1 - t_1} 1_{[x_1(t) \geq x_1(t_1), t \in [t_1, t_1 + l_2]]} \leq r_1(t_1)$, we just have to estimate from above $\mathbb{E}_{2s}(\sum_{t_1 \leq 2s'_1} r_1(t_1))$. Let us fix the value $0 \leq r_1 \leq 2s$. Then $y(t) = x(t + t_1) - x(t_1), 0 \leq t \leq r_1$, is a Dyck trajectory. Also, gluing the parts of the trajectory $x(t)$ corresponding to the time intervals $0 \leq t \leq t_1$ and $t_1 + r_1 \leq t \leq 2s$, one obtains a new Dyck trajectory of the length $2s - r_1$ which we

denote by $z(t)$. In other words, $z(t) = x(t)$, $0 \leq t \leq t_1$, and $z(t) = x(t + r_1)$, $t_1 \leq t \leq 2s - r_1$. One can choose the trajectory $z(\cdot)$ in at most $T_{0,2s-r_1} \leq \text{const}2^{2s-r_1} \times (2s - r_1)^{-3/2}$ ways. One can choose the instant t_1 in at most $2s - r_1$ ways. Finally, one can choose the trajectory $y(\cdot)$ in at most $T_{0,r_1} \leq \text{const}2^{r_1}r_1^{-3/2}$ ways. As a result,

$$\begin{aligned} \mathbb{E}_{2s} \left(\sum_{t_1 \leq 2s_1} r_1(t_1) \right) &\leq T_{0,2s}^{-1} \sum_{0 < r_1 < 2s} (2s - r_1) \text{const}^2 2^{2s-r_1} (2s - r_1)^{-3/2} 2^{r_1} r_1^{-3/2} r_1 \\ &\leq \text{Consts}^{3/2} \sum_{0 < r_1 < 2s} (2s - r_1)^{-1/2} r_1^{-1/2} \\ &\leq 2\text{Consts}^{3/2} \int_0^1 (1 - x)^{-1/2} x^{-1/2} dx. \end{aligned} \tag{44}$$

As always in this paper, the actual value of $\text{Const} > 0$ may change from line to line.

The general case (20) can be proven by the mathematical induction on $I'_i \geq 1$. We have to estimate from above

$$\mathbb{E}_{2s} \left(\sum_{0 \leq t_1 < t_2 < \dots < t_{I'_i} \leq 2s'_i} \right) \prod_{i=1}^{I'_i} r_i(t_i). \tag{45}$$

Let us define k so that $k + 1 = \max\{i : [t_i, t_i + r_i] \subset [t_1, t_1 + r_1]\}$. We apply the induction assumption to two sums:

- (i) over $t_2 \leq t_2 \leq \dots t_{k+1}$ with respect to the Dyck trajectory $y(\cdot)$, where $y(t) = x(t + t_1) - x(t_1)$, $t \in [0, r_1]$ and
- (ii) over $t_{k+2} \leq \dots t_{I'_i+1}$ with respect to the Dyck trajectory $z(\cdot)$ where $z(t) = x(t)$, $0 \leq t \leq t_1$, and $z(t) = x(t + r_1)$, $t_1 \leq t \leq 2s - r_1$. We arrive at the following sum

$$\begin{aligned} &s^{3/2} \sum_{k=0}^{I'_i-1} \sum_{0 < r_1 < 2s} r_1 (\text{Const}r_1^{3/2})^k \frac{1}{r_1^{3/2}} (2s - r_1) (\text{Const}(2s - r_1)^{3/2})^{(I'_i-k-1)} \\ &\quad \times \frac{1}{(2s - r_1)^{3/2}} \\ &\leq \text{Const}^{I'_i-1} (2s)^{I'_i} \text{const} \sum_{k=0}^{I'_i-1} \int_0^1 x^{3k/2-1/2} (1 - x)^{3(I'_i-1-k)/2-1/2} dx. \end{aligned} \tag{46}$$

The last sum in (46) is the sum of Beta functions

$$\begin{aligned} & \sum_{k=0}^{I_1'-1} B(3k/2 + 1/2, 3(I_1' - 1 - k)/2 + 1/2) \\ &= \sum_{k=0}^{I_1'-1} \frac{\Gamma(\frac{3}{2}k + \frac{1}{2}) \Gamma(\frac{3}{2}(I_1' - 1 - k) + \frac{1}{2})}{\Gamma(3(I_1' - 1) + 1)} \end{aligned}$$

and is bounded by the properties of the Beta and Gamma functions.

ACKNOWLEDGMENTS

Research of the second author was supported in part by the NSF grant DMS-0405864.

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